

Lattice Models of Quantum Gravity

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Abstract. In Standard Regge Calculus the quadratic link lengths of the considered simplicial manifolds vary continuously, whereas in the Z_2 -Regge Model they are restricted to two possible values. The goal is to determine whether the computationally more easily accessible Z_2 model retains the characteristics of standard Regge theory. We study both models in two dimensions employing the same functional integration measure and determine their phase structure by Monte Carlo simulations and mean-field theory.

Starting point for both Standard Regge Calculus (SRC) and the Z_2 -Regge Model (Z_2 RM) is Regge's discrete description of General Relativity in which space-time is represented by a piecewise flat, simplicial manifold: the Regge skeleton.^{1,2} The beauty of this procedure is that it works for any space-time dimension d and for metrics of arbitrary signature. The Einstein-Hilbert action translates into

$$I(q) = \lambda \sum_{s^d} V(s^d) - 2\beta \sum_{s^{d-2}} \delta(s^{d-2}) V(s^{d-2}) \quad , \quad (1)$$

with the quadratic edge lengths q describing the dynamics of the lattice, λ being the cosmological constant, and β the bare Planck mass squared. The first sum runs over all d -simplices s^d of the simplicial complex and $V(s^d)$ is the d -volume of the indicated simplex. The second term describes the curvature of the lattice, that is concentrated on the $(d-2)$ -simplices leading to deficit angles $\delta(s^{d-2})$, and is proportional to the integral over the curvature scalar in the classical Einstein-Hilbert action of the continuum theory.

In two dimensions this is easily illustrated by choosing a triangulation of the surface under consideration. Quantization of SRC proceeds by evaluating the

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Euclidean path integral

$$Z = \left[\prod_l \int_0^\infty dq_l q_l^{-m} \mathcal{F}(\{q_l\}) \right] \exp(-\lambda \sum_t A_t + 2\beta \sum_i \delta_i) . \quad (2)$$

In principle the functional integration should extend over all metrics on all possible topologies, but, as is usually done, we restrict ourselves to one specific topology, the torus. In two dimensions the sum over the deficit angles per vertex δ_i representing the scalar curvature corresponds due to the Gauss-Bonnet theorem to a vanishing Euler characteristic $2\pi\chi = \sum_i \delta_i = 0$. Consequently the action in the exponent of Eq. (2) consists only of a cosmological constant λ times the sum over all triangle areas A_t . The path-integral approach suffers from a non-uniqueness of the integration measure and it is even claimed that the true measure is of non-local nature.³ We used as a trial functional integration measure the expression within the square brackets of Eq. (2) with $m \in \mathbb{R}$ permitting to investigate a 1-parameter family of measures. The function \mathcal{F} constrains the integration to those configurations of link lengths which do not violate the triangle inequalities.

Although SRC can be efficiently vectorized for large scale computing, it is still a very time demanding enterprise. One therefore seeks for suitable approximations which will simplify the SRC and yet retain most of its universal features. The Z_2 RM could be such a desired simplification.⁴ Here the quadratic link lengths of the simplicial complexes are restricted to take on only the two values

$$q_l = 1 + \epsilon \sigma_l , \quad 0 < \epsilon < \epsilon_{max} , \quad \sigma_l = \pm 1 , \quad (3)$$

in close analogy to the ancestor of all lattice models, the Ising-Lenz model. Thus the area of a triangle with edges q_1, q_2, q_l is expressed as

$$\begin{aligned} A_t &= \frac{1}{2} \left| \begin{array}{cc} q_1 & \frac{1}{2}(q_1 + q_2 - q_l) \\ \frac{1}{2}(q_1 + q_2 - q_l) & q_2 \end{array} \right|^{\frac{1}{2}} = \\ &= c_0 + c_1(\sigma_1 + \sigma_2 + \sigma_l) + c_2(\sigma_1\sigma_2 + \sigma_1\sigma_l + \sigma_2\sigma_l) + c_3\sigma_1\sigma_2\sigma_l . \end{aligned} \quad (4)$$

The coefficients c_i depend on ϵ only and impose the condition $\epsilon < \frac{3}{5} = \epsilon_{max}$ in order to have real and positive triangle areas,⁴ i.e. $\mathcal{F} = 1$ for all possible configurations. The measure $\prod_l \int dq_l q_l^{-m}$ in Eq. (2) is replaced by

$$\sum_{\sigma_l = \pm 1} \exp[-m \sum_l \ln(1 + \epsilon \sigma_l)] = \sum_{\sigma_l = \pm 1} \exp[-N_1 m_0(\epsilon) - \sum_l m_1(\epsilon) \sigma_l] , \quad (5)$$

where N_1 is the total number of links, $m_0 = -\frac{1}{2}m\epsilon^2 + O(\epsilon^4)$, $m_1 = m[\epsilon + \frac{1}{3}\epsilon^3 + O(\epsilon^5)] = mM$ and $M = \sum_{i=1}^\infty \frac{\epsilon^{2i-1}}{2i-1}$. Hence the path integral Eq. (2) translates for the Z_2 RM into

$$Z = \sum_{\sigma_l = \pm 1} J \exp\left\{-\sum_l (2\lambda c_1 + m_1)\sigma_l - \lambda \sum_t [c_2(\sigma_1\sigma_2 + \sigma_1\sigma_l + \sigma_2\sigma_l) + c_3\sigma_1\sigma_2\sigma_l]\right\}, \quad (6)$$

with an unimportant constant J . If we view σ_l as a spin variable and assign it to the corresponding link l of the triangulation then Z reads as the partition function of a spin system with two- and three-spin nearest neighbour interactions on a Kagomé lattice. A particular simple form of Eq. (6) is obtained if $m_1 = -2\lambda c_1$ and therefore

$$m = \frac{-2\lambda c_1}{M} , \quad (7)$$

which is henceforth used for the measure in the Z_2 RM as well as in SRC. We set the parameter $\epsilon = 0.5$ in the following.

To compare both models we examined the quadratic link lengths and the area fluctuations on the simplicial lattice. Furthermore the Liouville mode is of special interest because it represents the only degree of freedom of pure 2d-gravity. The discrete analogue of the continuum Liouville field $\varphi(x) = \ln \sqrt{g(x)}$ is defined by

$$\phi = \frac{1}{A} \sum_i \ln A_i , \quad A_i = \frac{1}{3} \sum_{t \supset i} A_t , \quad (8)$$

where A_i is the area element of site i and A the total area.⁵ Additionally we are interested in the squared curvature defined by

$$R^2 = \sum_i \frac{\delta_i^2}{A_i} . \quad (9)$$

Figure 1 displays the corresponding expectation values as a function of the cosmological constant λ measured from 100k Monte Carlo sweeps after thermalization on simplicial lattices with 16×16 vertices.

Within the SRC the area increases with decreasing λ in perfect agreement with the scaling relation

$$\langle A \rangle = N_1 \frac{1-m}{\lambda} . \quad (10)$$

One also expects that $\langle q \rangle$ will increase as λ tends to zero. Actually we observe that the system thermalizes extremely slowly for very small λ and therefore display only statistically reliable data points for $\lambda \geq 1$ in the plots on the l.h.s. of Fig. 1. The Liouville field $\langle \phi \rangle$ behaves accordingly, and the expectation value of R^2 increases with λ .

Whereas the SRC becomes ill-defined for negative couplings λ , the Z_2 RM as an effective spin system is well-defined for all values of the cosmological constant. The phase transition the Z_2 RM undergoes at $\lambda_c \approx -11$ can be viewed as the relic of the transition from a well- to an ill-defined regime of SRC. A mean-field calculation for the Z_2 RM to extract the critical cosmological constant leads to $\lambda_c = -7.012$ and gives evidence for a (weak) first-order phase transition. It is a well-known property of mean-field theory to underestimate the critical coupling. The ratio between the mean-field and the numerical value of λ_c is in the same order of magnitude as for the exactly solvable, two-dimensional Ising model. A first-order phase transition would prevent to gain important information about the continuum theory, however, the true nature of the phase transition still remains to be determined.

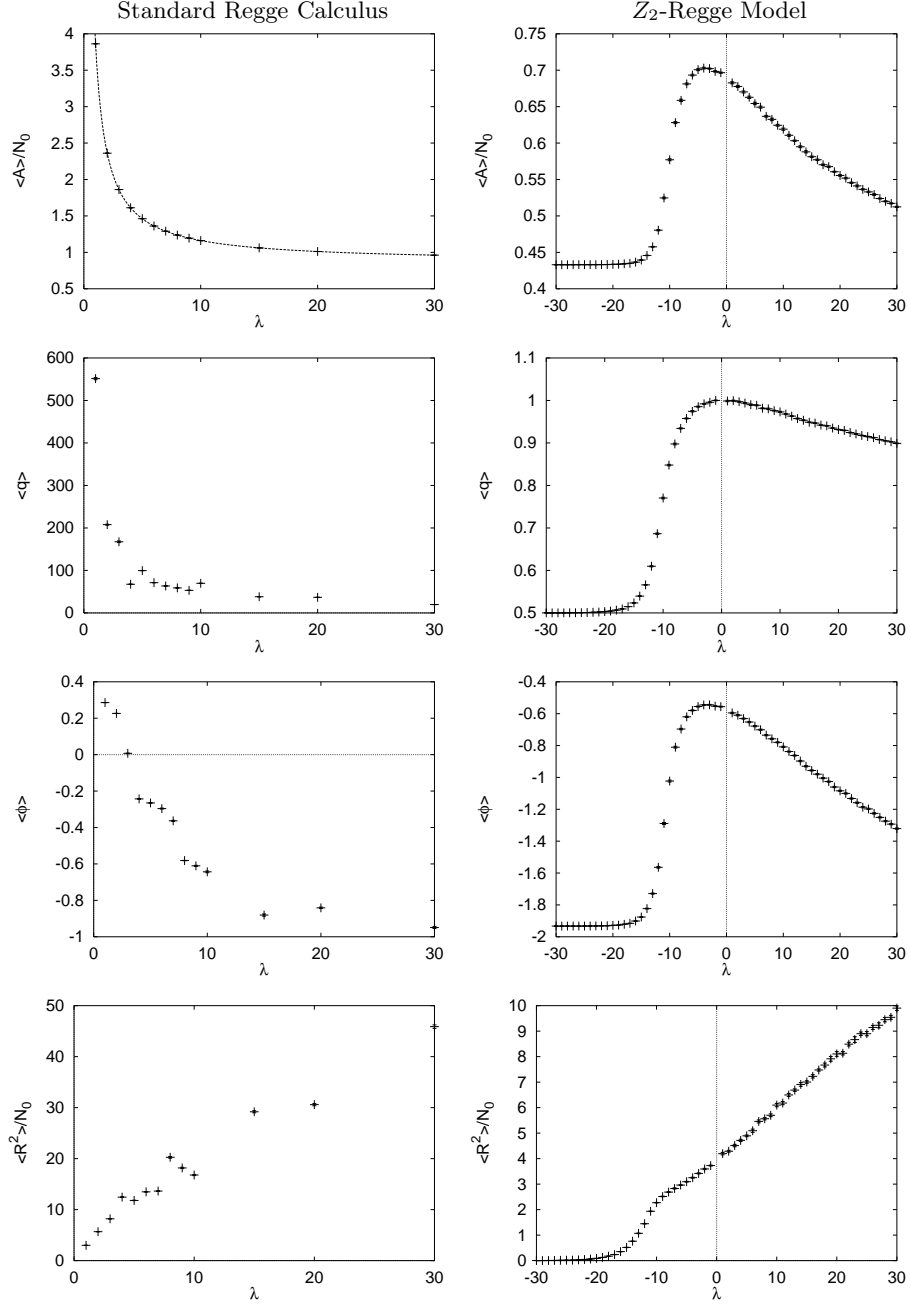


Fig. 1. Expectation values of the area A , the average squared link length q , the Liouville field ϕ , and the squared curvature R^2 as a function of the cosmological constant λ for the Standard Regge Calculus (left plots) and the Z_2 -Regge Model (right plots). N_0 is the total number of vertices.

Another interesting quantity to consider would be the Liouville susceptibility

$$\chi_\phi = \langle A \rangle [\langle \phi^2 \rangle - \langle \phi \rangle^2] . \quad (11)$$

From continuum field theory it is known that for fixed total area A the susceptibility scales according to

$$\ln \chi_\phi(L) \stackrel{L \rightarrow \infty}{\sim} c + (2 - \eta_\phi) \ln L , \quad (12)$$

with $L = \sqrt{A}$ and the Liouville field critical exponent $\eta_\phi = 0$. This has indeed been observed for SRC with the dq/q scale invariant measure and fixed area constraint.⁵ It is, however, a priori not clear if this feature will persist in the present model due to the fluctuating area and the non-scale invariant measure. This point is presently under investigation.

To conclude, physical observables like the Liouville field and the squared curvature behave similar in the Z_2 RM and SRC for the bare coupling $\lambda > \lambda_c$. The phase transition of the Z_2 RM in the negative coupling regime is interpreted as the remnant of the $\lambda = 0$ singularity of SRC. There remains the interesting question if by allowing for more than two link lengths the phase transition of such extended Z_2 RM approaches that of SRC. Then the situation might resemble the more involved four-dimensional case where one has to deal with 10 edges per simplex and the nontrivial Einstein-Hilbert action $\sum_{t \supset i} \delta_t A_t$ with 50 triangles t per vertex i in Eq. (1). Thus the action $I(q)$ takes on a large variety of values already for Z_2 RM and therefore SRC can be approximated more accurately.⁶

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